Spring 2019 Math 245 Final Exam

Please read the following directions:

Please write legibly, with plenty of white space. Please fill out the box above as legibly as possible. Please fit your answers in the designated areas. To get credit, you must also show adequate work to justify your answers. If unsure, show the work. All problems are worth 5-10 points. You may use a single page of notes, but no calculators or other aids. This exam will begin at 10:30 and will end at 12:30; pace yourself accordingly. Please remain quiet to ensure a good test environment for others. Good luck!

REMINDER: Use complete sentences.

Problem 1. Carefully define the following terms:

a. De Morgan's Law theorem (for propositions)

b. De Morgan's Law theorem (for sets)

c. Modus Ponens semantic theorem

Problem 2. Carefully define the following terms:

a. Direct Proof theorem

b. big O

c. symmetric difference

Problem 3. Prove that for all real x, we have $|x-1|+|x+2| \geq 3$.

REMINDER: Use complete sentences.

Problem 4. Carefully define the following terms:

a. power set

b. partition

c. Cartesian product

Problem 5. Carefully define the following terms:

a. set of departure

b. irreflexive

c. range (or image)

Problem 6. Without using the Classification Theorem, prove that $a_n \neq O(2^n)$, for $a_n = 3^n$.

Problem 7. Let p, q be propositions. Prove that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

Problem 8. Let p, q, r be propositions. Without truth tables, prove $(p \to q) \wedge (p \to r) \vdash p \to (q \wedge r)$.

Problem 9. Prove or disprove the proposition $\forall x \in \mathbb{N}, \exists y \in \mathbb{R}, x < y < 2x$.

Problem 10. Use induction to prove that, for every $n \in \mathbb{N}$ with $n \geq 3$, we have $n^2 \geq 2n + 1$.

Problem 11. Let S, T, U be sets, with $S \subseteq T \subseteq U$. Prove that $T^c \subseteq S^c$.

Problem 12. Let R be a relation on set S. Suppose that R is both symmetric and antisymmetric. Prove that $R \subseteq R_{diagonal}$.

Problem 13. Define relation R on $\mathbb{Z}$ via $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 4|(a + b)\}$. Prove or disprove that R is an equivalence relation.

Problem 14. Find all integers $x \in [0, 24)$ satisfying the modular equation $10x \equiv 8 \pmod{24}$.

Problem 15. Let R be a partial order on S, and $T \subseteq S$. Suppose that $a, a' \in T$ are both greatest in T. Prove that $a = a'$.

The last five problems, 16-20, all concern relations on the following ground set:

$$S' = \{\text{polynomials with integer coefficients, in variable } x\}$$

Five sample elements of S' are $r(x) = x^2 - 2x + 1, s(x) = 7x + 3, t(x) = 7x, u(x) = 3, v(x) = 0$.

Problem 16. Consider the equivalence relation $R_1 = \{(f(x), g(x)) : f(2) = g(2)\}$ on S'. Give any three elements of $[w(x)]$, for $w(x) = x + 1$. Note: S' is defined at the top of this page.

Problem 17. Consider the relation $R_2 = \{(f(x), g(x)) : \forall y \in \mathbb{R}, f(y) \leq g(y)\}$ on S'. Prove that R_2 is a partial order. Note: S' is defined at the top of this page.

Problem 18. Consider the partial order $R_2 = \{(f(x), g(x)) : \forall y \in \mathbb{R}, f(y) \leq g(y)\}$ on S'. Find an antichain of size 3 (be sure to fully justify). Note: S' is defined at the top of the previous page.

Problem 19. Consider the relation $R_3 = \{(f(x), g(x)) : \forall y \in \mathbb{R}, g(y) + 2f(y) = 1\}$ on S'. Prove that R_3 is a function. Note: S' is defined at the top of the previous page.

Problem 20. Consider the function $R_3 = \{(f(x), g(x)) : \forall y \in \mathbb{R}, g(y) + 2f(y) = 1\}$ on S'. Prove or disprove that R_3 is a bijection. Note: S' is defined at the top of the previous page.